

Rui Miguel Saramago

Connected Hopf corings and their Dieudonné counterparts

Received: 1 October 2013 / Accepted: 15 June 2014 / Published online: 11 July 2014
© The Author(s) 2014. This article is published with open access at Springerlink.com

Abstract We define coring objects in the category of algebras over a perfect field of characteristic p (with connected underlying Hopf algebra) and the corresponding notion for Dieudonné modules, and prove the equivalence of the two resulting categories, extending thus the methods of Dieudonné theory for Hopf rings from Ravenel (Reunión Sobre Teoría de Homotopía, volume 1 of Serie notas de matemática y simposia, 177–194, 1975), Schoeller (Manusc Math 3:133–155, 1970), Goerss (Homotopy invariant algebraic structures: a conference in honor of J. Michael Boardman, 115–174, 1999) and Saramago (Dieudonné theory for ungraded and periodically graded Hopf rings, Ph.D. thesis, The Johns Hopkins University, 2000).

Mathematics Subject Classification 16W30 · 57T05 · 18E10

المخلص

نعرف حلقات مرافقة في فئة الجبريات على حقل كامل مميزه p (بحيث تكون جبرية هوبف المتضمنة مترابطة)، والمفهوم المقابل لحلقات ديوندي، ونثبت تكافؤ الفئتين الناتجتين، ممددين بذلك طرق نظرية ديوندي لحلقات هوبف من [4]، و [10] و [3] و [7].

1 Introduction

Dieudonné theory permits the construction of functors giving equivalences between certain categories of Hopf algebras and corresponding categories of modules (over specific rings), called their Dieudonné modules. Such functors have been defined for some objects in the categories of graded Hopf algebras [4, 10], ungraded Hopf algebras [1] and periodically graded Hopf algebras [6]. Additionally, functors from categories of Hopf rings have also been devised: for the graded case [3] and the ungraded and periodically graded cases [2, 7, 8].

To prove such equivalences, it is usual to give a universal bilinear product in the categories of Hopf algebras and also in the corresponding categories of Dieudonné modules. One proves that these two products correspond functorially. This then reflects on an equivalence between Hopf rings and Dieudonné rings.

In this paper, we define what such a correspondence should be for *Hopf corings*, that is, algebras furnished with two coproducts that are related by a form of *codistributivity*. We present universal cobilinear coproducts of connected Hopf algebras and the corresponding notion for Dieudonné modules. This allows us to extend Dieudonné theory for Hopf corings, giving the equivalence of these categories.

R. M. Saramago (✉)
Departamento de Matemática, Centro de Análise Matemática, Geometria e Sistemas Dinâmicos,
Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisbon, Portugal
E-mail: saramago@math.ist.utl.pt



2 Bilinear products of Hopf algebras over a perfect field of characteristic p

Given a Hopf algebra H over a perfect field k of characteristic p , the coproduct of an element $x \in H$ will be written as $\psi(x) = \sum x^{(1)} \otimes x^{(2)}$ and its co-unit will be denoted by $\epsilon : H \rightarrow k$. Its unit map will be denoted by $\eta : k \rightarrow H$. The Frobenius morphism $F : H \rightarrow H$ is given by $F(x) = x^p$, and the Verschiebung will be denoted by $V : H \rightarrow H$ (dual to the Frobenius in the dual algebra). We will write $*$ for the algebra product.

For the results in Sect. 4, we will need the definition of induced primitives from [9]. In characteristic 2, a *first-order induced primitive* element of H (relative to the primitive q) is any $x \in H$ that does $\psi(x) = 1 \otimes x + x \otimes 1 + q \otimes q$. The difference between two induced primitives (relative to the same q) is always a primitive element. Any first order induced primitive relative to q will be denoted by \hat{q} (or $q^{(1)}$).

An induced primitive behaves, thus, as a $q^2/2$ does whenever the characteristic of the base field is not 2.

A *second-order induced primitive* element of H (in characteristic 2, relative to the primitive q) is any x such that $\psi(x)$ contains a term $\hat{q} \otimes \hat{q}$ (for some first order primitive \hat{q} relative to q) and additionally just the extra terms that the definition of Hopf algebra imposes. Specifically, if $\psi(x)$ contains $\hat{q} \otimes \hat{q}$, then $(1 \otimes \psi)(\psi(x))$ contains $\hat{q} \otimes q \otimes q$, so $(\psi \otimes 1)(\psi(x))$ contains the same, and so $\psi(x)$ contains a $a \otimes q$ with $\psi(a)$ featuring $\hat{q} \otimes q$. Following this reasoning through, one gets that any second-order induced primitive x relative to q must actually make

$$\psi(x) = 1 \otimes x + x \otimes 1 + q \otimes q \hat{q} + q \hat{q} \otimes q + q^2 \otimes \hat{q} + \hat{q} \otimes q^2 + \hat{q} \otimes \hat{q}$$

This is the minimum number of terms necessary for $\psi(x)$ to feature a $\hat{q} \otimes \hat{q}$, and we denote by $q^{(2)}$ any second-order induced primitive relative to q (and to the \hat{q} chosen.)

Denote by $q^{(m)}$ an *induced primitive of order m* , in characteristic 2, relative to the primitive q , which is defined as any x in H whose coproduct contains a term $q^{(m-1)} \otimes q^{(m-1)}$ plus all the minimum required terms that the definition of Hopf algebra imposes.

If the characteristic is an odd prime r , a *first-order induced primitive* element \hat{q} (or $q^{(1)}$) (relative to the primitive q) is any $x \in H$ whose coproduct contains $q \otimes q^{r-1}$ and also just all the terms that the definition of Hopf algebra imposes, and an *induced primitive of order m* (relative to the primitive q), is as any $x \in H$ whose coproduct contains $q^{(m-1)} \otimes [q^{(m-1)}]^{r-1}$ and additionally just all the terms required by the definition of Hopf algebra.

In any case, the Verschiebung acts on induced primitives by $V(q^{(m)}) = q^{(m-1)}$ (where $q^{(m-1)}$ is the order $m-1$ primitive relative to q that was chosen in the definition of the $q^{(m)}$ given.)

Let H_1 , H_2 and K be Hopf algebras over a commutative ring R . A morphism of coalgebras

$$\phi : H_1 \otimes H_2 \rightarrow K$$

is called a *bilinear map of coalgebras* if we have:

- (1) $\phi(xy, z) = \sum \phi(x, z^{(1)}) \phi(y, z^{(2)})$
- (2) $\phi(x, yz) = \sum \phi(x^{(1)}, y) \phi(x^{(2)}, z)$
- (3) $\phi(x, 1) = \epsilon(x) \cdot 1$
- (4) $\phi(1, y) = \epsilon(y) \cdot 1$

These relations can be viewed as reflecting the commutativity of some corresponding diagrams. For example, (1) states the commutativity of

$$\begin{array}{ccccc} H_1 \otimes H_1 \otimes H_2 & \xrightarrow{* \otimes 1} & H_1 \otimes H_2 & \xrightarrow{\phi} & K \\ \downarrow 1 \otimes \psi & & & & \uparrow * \\ H_1 \otimes H_1 \otimes H_2 \otimes H_2 & \xrightarrow{1 \otimes sw \otimes 1} & H_1 \otimes H_2 \otimes H_1 \otimes H_2 & \xrightarrow{\phi \otimes \phi} & K \otimes K \end{array}$$

(Here, sw is the switch map.)

Hopf algebras have universal bilinear products [3]. This is defined, for each pair of Hopf algebras H_1 and H_2 , as the unique Hopf algebra $H_1 \boxtimes H_2$ together with a bilinear map

$$\gamma : H_1 \otimes H_2 \rightarrow H_1 \boxtimes H_2$$



such that for any bilinear map $H_1 \otimes H_2 \rightarrow K$ there exists a unique Hopf algebra map $H_1 \boxtimes H_2 \rightarrow K$ that makes the following diagram commute:

$$\begin{array}{ccc} H_1 \otimes H_2 & \xrightarrow{\gamma} & H_1 \boxtimes H_2 \\ \downarrow & \swarrow & \\ K & & \end{array}$$

This universal bilinear product is constructed as a quotient of the symmetric algebra $S(H_1 \otimes H_2)$ on $H_1 \otimes H_2$. This symmetric algebra has a coproduct given by the requirement that the inclusion $H_1 \otimes H_2 \rightarrow S(H_1 \otimes H_2)$ be a map of coalgebras. That is, for $a \in H_1$ and $b \in H_2$, we have

$$\psi(a \otimes b) = \sum (a^{(1)} \otimes b^{(1)}) \otimes (a^{(2)} \otimes b^{(2)})$$

Consider the ideal J in $S(H_1 \otimes H_2)$ generated by the elements

- (1) $(xy) \otimes z - \sum (x \otimes z^{(1)}) \otimes (y \otimes z^{(2)})$
- (2) $x \otimes (yz) - \sum (x^{(1)} \otimes y) \otimes (x^{(2)} \otimes z)$
- (3) $x \otimes 1 - \epsilon(x) \cdot 1$
- (4) $1 \otimes y - \epsilon(y) \cdot 1$

Proposition 2.1 [2] *The algebra $S(H_1 \otimes H_2)/J$ has a structure of Hopf algebra that realizes the bilinear product $H_1 \boxtimes H_2$ of H_1 and H_2 .*

The universal bilinear product γ is related to the Frobenius and Verschiebung in the following way [2].

Given $a \in H_1$ and $b \in H_2$, we have:

- (1) $V(\gamma(a, b)) = \gamma(Va, Vb)$
- (2) $F(\gamma(Va, b)) = \gamma(a, Fb)$
- (3) $F(\gamma(a, Vb)) = \gamma(Fa, b)$

These are the properties used in the proof of the equivalence between categories of Hopf rings and Dieudonné rings.

The construction of $H_1 \boxtimes H_2$ permits us to give a straight definition of Hopf rings. A *commutative Hopf ring* over a commutative ring R is a Hopf algebra H over R together with a commutative map $\phi : H \boxtimes H \rightarrow H$ that is associative: $\phi(\phi \boxtimes 1) = \phi(1 \boxtimes \phi)$.

This definition is equivalent to saying there has to be a circle product $\circ : H \otimes H \rightarrow H$, which is a map of *coalgebras*, satisfying convenient distributivity properties with respect to the algebra product [5], as given by the following commutative diagrams

$$\begin{array}{ccccc} H \otimes H \otimes H & \xrightarrow{\psi \otimes 1} & H \otimes H \otimes H \otimes H & \xrightarrow{1 \otimes sw \otimes 1} & H \otimes H \otimes H \otimes H \\ \downarrow 1 \otimes * & & & & \downarrow \circ \otimes \circ \\ H \otimes H & \xrightarrow{\circ} & H & \xleftarrow{*} & H \otimes H \end{array}$$

$$\begin{array}{ccccc} H \otimes H \otimes H & \xrightarrow{1 \otimes \psi} & H \otimes H \otimes H \otimes H & \xrightarrow{1 \otimes sw \otimes 1} & H \otimes H \otimes H \otimes H \\ \downarrow * \otimes 1 & & & & \downarrow \circ \otimes \circ \\ H \otimes H & \xrightarrow{\circ} & H & \xleftarrow{*} & H \otimes H \end{array}$$

(Here, $*$ represents the algebra product and sw is the switch map.)



3 Dieudonné ring theory

Let \mathcal{HA} be the category of ungraded connected Hopf algebras over \mathbb{F}_p , p a prime (Additional details for the concepts at hand can be found in [8]).

Given a sequence of indeterminates $\{x_i\}$, we consider the Hopf algebras $H(n) = \mathbb{F}_p[x_0, \dots, x_n]$ for $n \geq 0$. We have Hopf algebra maps $\alpha : H(n) \rightarrow H(n+1)$ (given by inclusion) and $\bar{V} : H(n+1) \rightarrow H(n)$ (defined by $\bar{V}(x_i) = x_{i-1}$ for $i > 0$ and $\bar{V}(x_0) = 0$). $\bar{V}\alpha$ gives the Verschiebung on each $H(n)$. We have thus a sequence

$$\cdots \longrightarrow H(n+1) \xrightarrow{\bar{V}} H(n) \xrightarrow{\bar{V}} H(n-1) \longrightarrow \cdots$$

For each ungraded connected Hopf algebra H , this sequence induces a sequence of \mathbb{F}_p -modules

$$\cdots \rightarrow \text{Hom}_{\mathcal{HA}}(H(n-1), H) \xrightarrow{\hat{V}} \text{Hom}_{\mathcal{HA}}(H(n), H) \xrightarrow{\hat{V}} \text{Hom}_{\mathcal{HA}}(H(n+1), H) \rightarrow \cdots$$

where each \hat{V} is given by composition with \bar{V} on the left.

Consider now the \mathbb{F}_p -module $DH = \text{colim}_n \text{Hom}_{\mathcal{HA}}(H(n), H)$. Composing on the right with the Verschiebung $V : H \rightarrow H$ and the Frobenius $F : H \rightarrow H$ gives maps $V : DH \rightarrow DH$ and $F : DH \rightarrow DH$. We have $FV = VF = p$. Given a Hopf algebra $H \in \mathcal{HA}$, we define its *Dieudonné module* as the module $DH = \text{colim}_n \text{Hom}_{\mathcal{HA}}(H(n), H)$ together with the homomorphisms $V : DH \rightarrow DH$ and $F : DH \rightarrow DH$ given above.

Since any $H \in \mathcal{HA}$ is connected, its coaugmentation filtration exhausts it and, moreover, if we write $\psi(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$ for each $x \in F_q H$, then all the x' and x'' that appear in the expression are in those $F_{q'} H$ that have $q < q'$. Thus, the Verschiebung on such Hopf algebras is eventually zero. This carries over to DH , where we have that for each $x \in DH$ there must exist an $n \geq 0$ such that $V^n x = 0$.

Definition 3.1 The category \mathcal{DM} of *ungraded connected Dieudonné modules* has as objects modules M over the ring $R = \mathbb{Z}_p[V, F]/(VF = FV = p)$ such that for each $x \in M$ there exists an $n \geq 0$ with $V^n x = 0$.

Here, \mathbb{Z}_p are the p -adic integers.

The above considerations give us a functor $D : \mathcal{HA} \rightarrow \mathcal{DM}$ that takes a Hopf algebra $H \in \mathcal{HA}$ and produces its Dieudonné module $DH = \text{colim}_n \text{Hom}_{\mathcal{HA}}(H(n), H)$.

Theorem 3.2 [3] *The functor $D : \mathcal{HA} \rightarrow \mathcal{DM}$ has a right adjoint $U : \mathcal{DM} \rightarrow \mathcal{HA}$, and the pair (D, U) forms an equivalence of categories.*

One defines a bilinear map for R -modules M, N and L as a map $g : M \otimes N \rightarrow L$ that satisfies:

- (1) $Vg(m \otimes n) = g(Vm \otimes Vn)$
- (2) $Fg(Vm \otimes n) = g(m \otimes Fn)$
- (3) $Fg(m \otimes Vn) = g(Fm \otimes n)$

for every $m \in M$ and $n \in N$.

(This is a similar definition to that made for Hopf algebras.)

Two Dieudonné modules M and N also have a universal bilinear product, given by a Dieudonné module $M \boxtimes N$ together with a bilinear map $M \otimes N \rightarrow M \boxtimes N$ that is universal with respect to all bilinear maps [3], [2].

The following result gives the fundamental equivalence between the category of ungraded connected Hopf rings and that of *connected Dieudonné rings*, which are ungraded connected Dieudonné modules together with a product compatible with V and F .

Theorem 3.3 [2,3] *The category \mathcal{DM} together with the bilinear product \boxtimes is equivalent to the category of ungraded connected Hopf rings.*



4 Cobilinear coproducts of connected Hopf algebras over a perfect field of characteristic p

Next, we build for algebras a theory corresponding to what was done for coalgebras, defining cobilinear maps, universal cobilinear maps and the notion of a Hopf coring.

We start with Hopf algebras over a commutative ring R . Let H_1 , H_2 and K be such Hopf algebras. A morphism of algebras

$$\phi : K \rightarrow H_1 \otimes H_2$$

is called a *cobilinear map* if the following diagrams commute:

$$\begin{array}{ccccc} K & \xrightarrow{\phi} & H_1 \otimes H_2 & \xrightarrow{\psi \otimes 1} & H_1 \otimes H_1 \otimes H_2 \\ \psi \downarrow & & & & \uparrow 1 \otimes * \\ K \otimes K & \xrightarrow{\phi \otimes \phi} & H_1 \otimes H_2 \otimes H_1 \otimes H_2 & \xrightarrow{1 \otimes sw \otimes 1} & H_1 \otimes H_1 \otimes H_2 \otimes H_2 \\ \\ K & \xrightarrow{\phi} & H_1 \otimes H_2 & \xrightarrow{1 \otimes \psi} & H_1 \otimes H_2 \otimes H_2 \\ \psi \downarrow & & & & \uparrow * \otimes 1 \\ K \otimes K & \xrightarrow{\phi \otimes \phi} & H_1 \otimes H_2 \otimes H_1 \otimes H_2 & \xrightarrow{1 \otimes sw \otimes 1} & H_1 \otimes H_1 \otimes H_2 \otimes H_2 \\ \\ K & \xrightarrow{\phi} & H_1 \otimes H_2 & \xrightarrow{1 \otimes \epsilon} & H_1 \otimes R \\ \psi \downarrow & & & & \uparrow \eta \otimes 1 \\ K \otimes K & \xrightarrow{\phi \otimes \phi} & H_1 \otimes H_2 \otimes H_1 \otimes H_2 & \xrightarrow{\epsilon \otimes \epsilon} & R \otimes R \\ \\ K & \xrightarrow{\phi} & H_1 \otimes H_2 & \xrightarrow{\epsilon \otimes 1} & R \otimes H_2 \\ \psi \downarrow & & & & \uparrow 1 \otimes \eta \\ K \otimes K & \xrightarrow{\phi \otimes \phi} & H_1 \otimes H_2 \otimes H_1 \otimes H_2 & \xrightarrow{\epsilon \otimes \epsilon} & R \otimes R \end{array}$$

The diagrams reflect the inversion of the diagrams one can construct to define the bilinearity of maps between coalgebras.

As for bilinear products, we can also describe cobilinear maps in terms of relations, but these do not come up as neatly as the above diagrams. Given $x \in K$ and $\phi : K \rightarrow H_1 \otimes H_2$ cobilinear, denote $\phi(x)$ by $\sum x_{(1)} \otimes x_{(2)}$ (recall that superscripts are reserved for coproducts.) Using this notation, we then get, for any $x \in K$:

- (1) $\sum (x_{(1)})^{(1)} \otimes (x_{(1)})^{(2)} \otimes x_{(2)} = \sum (x^{(1)})_{(1)} \otimes (x^{(2)})_{(1)} \otimes [(x^{(1)})_{(2)} (x^{(2)})_{(2)}]$
- (2) $\sum x_{(1)} \otimes (x_{(2)})^{(1)} \otimes (x_{(2)})^{(2)} = \sum [(x^{(1)})_{(1)} (x^{(2)})_{(1)}] \otimes (x^{(1)})_{(2)} \otimes (x^{(2)})_{(2)}$
- (3) $\sum x_{(1)} \otimes \epsilon_{H_2}(x_{(2)}) = \sum \eta_{H_1}(\epsilon_K(x^{(1)})) \otimes \epsilon_K(x^{(2)})$
- (4) $\sum \epsilon_{H_1}(x_{(1)}) \otimes x_{(2)} = \sum \epsilon_K(x^{(1)}) \otimes \eta_{H_2}(\epsilon_K(x^{(2)}))$

The universal cobilinear coproduct is defined, for each pair of Hopf algebras H_1 and H_2 , when it exists, as the unique Hopf algebra $H_1 \hat{\boxtimes} H_2$ together with a cobilinear map (of algebras)

$$\gamma : H_1 \hat{\boxtimes} H_2 \rightarrow H_1 \otimes H_2$$

that is universal with respect to all cobilinear maps; that is, such that for any cobilinear map $K \rightarrow H_1 \otimes H_2$ there exists a unique Hopf algebra map $K \rightarrow H_1 \hat{\boxtimes} H_2$ that makes the following diagram commute:

$$\begin{array}{ccc} H_1 \hat{\boxtimes} H_2 & \xrightarrow{\gamma} & H_1 \otimes H_2 \\ \uparrow & \nearrow & \\ K & & \end{array}$$

Theorem 4.1 Any two connected Hopf algebras over a perfect field have a universal cobilinear coproduct.

Proof Suppose first the field has characteristic zero. Any connected Hopf algebra over such a field is generated as an algebra by its primitive elements (see [9]). Thus, for any algebra map $\phi : K \rightarrow H_1 \otimes H_2$ (given connected Hopf algebras H_1, H_2 and K), the polynomial algebra generated by all elements $\phi(q)$, where q is a primitive in K , has a Hopf algebra structure given by putting all these generators as primitives. This Hopf algebra is moreover a connected Hopf algebra by construction.

Given two connected Hopf algebras H_1 and H_2 , consider then any Hopf algebra K and any cobilinear map $\phi : K \rightarrow H_1 \otimes H_2$. Write $P(K, \phi)$ for the polynomial algebra $P(\{\phi(q) : q \text{ is primitive}\})$. Give it a Hopf algebra structure by declaring $\phi(q)$ a primitive whenever q is a primitive. Then we define

$$H_1 \widehat{\otimes} H_2 = \coprod_{K, \phi} P(K, \phi),$$

the disjoint union of all the $P(K, \phi)$.

This has the obvious operations that make it a Hopf algebra, which is connected because it is generated by its primitives.

The map $H_1 \widehat{\otimes} H_2 \rightarrow H_1 \otimes H_2$ given on each generator $\phi(q)$ by $\phi(q)$ (viewed as an element in $H_1 \otimes H_2$) is a cobilinear map that moreover will be universal by construction: Given a Hopf algebra K and a cobilinear map $K \rightarrow H_1 \otimes H_2$, the very definition of $\widehat{\otimes}$ implies the existence of a unique map $K \rightarrow H_1 \widehat{\otimes} H_2$ making the corresponding diagram commute. Since the Hopf algebras are connected, the fact that the diagram commutes for primitives is enough to prove that it is truly a Hopf algebra map.

If the field has characteristic a prime, Saramago [9] says that any connected Hopf algebra over it is generated as an algebra by its primitive and induced primitives. In this case, write $P(K, \phi)$ for the polynomial algebra $P(\{\phi(q) : q \text{ is primitive or induced primitive}\})$. Its Hopf algebra structure is now obtained by declaring, for each primitive $q \in K$, that $\phi(q)$ is primitive and $[\phi(q)]^{(n)} = \phi(q^{(n)})$ whenever $q^{(n)}$ is an induced primitive of order n in K .

Build $H_1 \widehat{\otimes} H_2$ as before, and define the map $H_1 \widehat{\otimes} H_2 \rightarrow H_1 \otimes H_2$ similarly. Again, for each cobilinear $K \rightarrow H_1 \otimes H_2$ there exists a natural map $K \rightarrow H_1 \widehat{\otimes} H_2$. The corresponding diagram commutes for primitives and induced primitives of K , and the result follows from the fact that giving the values of a map on primitives and induced primitives is enough to define it completely whenever the Hopf algebra K is connected. \square

The definition of a universal cobilinear coproduct for connected Hopf algebras gives us a way to define what a connected Hopf coring should be. This is given as a connected Hopf k -algebra H (where k is a perfect field of characteristic p) together with a map $\phi : H \rightarrow H \widehat{\otimes} H$ that is *coassociative*, that is, such that $(\phi \widehat{\otimes} 1) \phi = (1 \widehat{\otimes} \phi) \phi$ as maps from H to $H \widehat{\otimes} H \widehat{\otimes} H$.

By the universal construction, this is equivalent to giving two maps of algebras (two "coproducts") $H \rightarrow H \otimes H$ that are related by a codistributivity.

Given a connected Hopf coring, its map $\phi : H \rightarrow H \widehat{\otimes} H$, when composed with the defining map $\gamma : H \widehat{\otimes} H \rightarrow H \otimes H$, gives an algebra map $\tilde{\phi} : H \rightarrow H \otimes H$.

This map $\tilde{\phi}$, together with the original coproduct ψ , satisfies a form of codistributivity. This comes from the inversion of the diagrams from Sect. 2 that deal with distributivity for Hopf rings.

We get then:

$$\begin{array}{ccccc}
 H \otimes H \otimes H & \xleftarrow{* \otimes 1} & H \otimes H \otimes H \otimes H & \xleftarrow{1 \otimes sw \otimes 1} & H \otimes H \otimes H \otimes H \\
 \uparrow 1 \otimes \psi & & & & \uparrow \tilde{\phi} \otimes \tilde{\phi} \\
 H \otimes H & \xleftarrow{\tilde{\phi}} & H & \xrightarrow{\psi} & H \otimes H
 \end{array}$$



$$\begin{array}{ccccc}
 H \otimes H \otimes H & \xleftarrow{1 \otimes *} & H \otimes H \otimes H \otimes H & \xleftarrow{1 \otimes sw \otimes 1} & H \otimes H \otimes H \otimes H \\
 \uparrow \psi \otimes 1 & & & & \uparrow \tilde{\phi} \otimes \tilde{\phi} \\
 H \otimes H & \xleftarrow{\tilde{\phi}} & H & \xrightarrow{\psi} & H \otimes H
 \end{array}$$

We can write the relations coming from these diagrams. Subscripts refer to the proper coproduct in each case, and for instance the first diagram yields:

$$\sum x_{\tilde{\phi}}^{(1)} \otimes (x_{\tilde{\phi}}^{(2)})_{\psi}^{(1)} \otimes (x_{\tilde{\phi}}^{(2)})_{\psi}^{(2)} = \sum (x_{\psi}^{(1)})_{\tilde{\phi}}^{(1)} (x_{\psi}^{(2)})_{\tilde{\phi}}^{(1)} \otimes (x_{\psi}^{(1)})_{\tilde{\phi}}^{(2)} \otimes (x_{\psi}^{(2)})_{\tilde{\phi}}^{(2)}$$

for each $x \in H$.

(Compare Eq. (2) on page 6.)

As for coassociativity, it comes from the following two diagrams.

$$\begin{array}{ccccc}
 (H \hat{\boxtimes} H) \hat{\boxtimes} H & \xrightarrow{\gamma} & (H \hat{\boxtimes} H) \otimes H & \xrightarrow{\gamma \otimes 1} & (H \otimes H) \otimes H \\
 \uparrow \phi \hat{\boxtimes} 1 & & \uparrow \phi \otimes 1 & \nearrow (\gamma \otimes 1) \circ (\phi \otimes 1) = \widetilde{\phi \otimes 1} & \\
 H \hat{\boxtimes} H & & H \otimes H & & \\
 \uparrow \phi & \nearrow \tilde{\phi} & & & \\
 H & & & &
 \end{array}$$

$$\begin{array}{ccccc}
 H \hat{\boxtimes} (H \hat{\boxtimes} H) & \xrightarrow{\gamma} & H \otimes (H \hat{\boxtimes} H) & \xrightarrow{1 \otimes \gamma} & H \otimes (H \otimes H) \\
 \uparrow 1 \hat{\boxtimes} \phi & & \uparrow 1 \otimes \phi & \nearrow (1 \otimes \gamma) \circ (1 \otimes \phi) = \widetilde{1 \otimes \phi} & \\
 H \hat{\boxtimes} H & & H \otimes H & & \\
 \uparrow \phi & \nearrow \tilde{\phi} & & & \\
 H & & & &
 \end{array}$$

The condition translates thus as

$$(\widetilde{\phi \otimes 1}) \circ \tilde{\phi} = (\widetilde{1 \otimes \phi}) \circ \tilde{\phi}$$

as maps from H to $H \otimes H \otimes H$.

Next, we deduce how the universal bilinear coproduct $\gamma : H_1 \hat{\boxtimes} H_2 \rightarrow H_1 \otimes H_2$ relates to the Frobenius and the Verschiebung.

For this, we first need a result that states what cobilinear maps $\phi : K \rightarrow H_1 \otimes H_2$ do on primitives and induced primitives of K .



Lemma 4.2 Let $\phi : K \rightarrow H_1 \otimes H_2$ be a cobilinear map.

(1) If $x \in K$ is a primitive element, then

$$\phi(x) = \sum_{\alpha, \beta} q_\alpha \otimes q_\beta$$

where all q_α and q_β are primitive.

(2) If $x \in K$ is an induced primitive of order n (in characteristic 2), then

$$\phi(x) = \sum_{\alpha, \beta} \sum_{i=0}^n [q_\alpha^{(n-i)}]^{2^i} \otimes [q_\beta^{(i)}]^{2^{n-i}}$$

where all q_α and q_β are primitive.

(3) If $x \in K$ is an induced primitive of order n (in odd characteristic r), then

$$\phi(x) = \sum_{\alpha, \beta} \sum_{i=0}^n [q_\alpha^{(n-i)}]^{2^i} \otimes [[q_\beta^{(i)}]^{r-1}]^{2^{n-i}}$$

where all q_α and q_β are primitive.

Proof (1) The first relation in the definition of cobilinear map (from page 6), whenever applied to a primitive x , gives

$$\sum (x_{(1)})^{(1)} \otimes (x_{(1)})^{(2)} \otimes x_{(2)} = \sum [1 \otimes x_{(1)} \otimes x_{(2)} + x_{(1)} \otimes 1 \otimes x_{(2)}]$$

and so $\psi(x_{(1)}) = 1 \otimes x_{(1)} + x_{(1)} \otimes 1$ for any $x_{(1)}$.

The second relation from page 6 gives that any $x_{(2)}$ is also primitive.

(2) We prove it by induction in n . If $n = 1$, then $\psi(x) = 1 \otimes x + x \otimes 1 + q \otimes q$, with q a primitive, and so the first relation from page 6 now becomes

$$\sum (x_{(1)})^{(1)} \otimes (x_{(1)})^{(2)} \otimes x_{(2)} = \sum [1 \otimes x_{(1)} \otimes x_{(2)} + x_{(1)} \otimes 1 \otimes x_{(2)} + q_{(1)} \otimes q_{(1)} \otimes q_{(2)}^2]$$

From this one gets that each $x_{(1)}$ is either primitive or a first-order induced primitive relative to $q_{(1)}$ (which is primitive by the first part of this lemma), and in this case $x_{(2)} = q_{(2)}^2$ (which in characteristic 2 is also a primitive). Using the second relation from page 6, we get a similar result if one switches $x_{(1)}$ and $x_{(2)}$, and so

$$\phi(x) = \sum x_{(1)} \otimes x_{(2)} = \sum_{\alpha, \beta} [q_\alpha^{(1)} \otimes q_\beta^2 + q_\beta^2 \otimes q_\alpha^{(1)}]$$

where all q_α and q_β are primitive.

If the result is true for $n - 1$, let x be an induced primitive of order n relative to the primitive q . Then $\psi(x)$ has a $q^{(n-1)} \otimes q^{(n-1)}$, and so the right side in the first relation from page 6 has a

$$(q^{(n-1)})_{(1)} \otimes (q^{(n-1)})_{(1)} \otimes [(q^{(n-1)})_{(2)}]^2$$

We know by induction that $(q^{(n-1)})_{(1)} \otimes (q^{(n-1)})_{(2)} = [q_\alpha^{(n-i)}]^{2^i} \otimes [q_\beta^{(i)}]^{2^{n-i}}$ for some i and some primitives q_α and q_β . We get then that a $x_{(1)}$ is either primitive or is a $[q_\alpha^{(n-i+1)}]^{2^i}$, and in this case $x_{(2)} = [q_\beta^{(i)}]^{2^{n-i+1}}$.

Using the second relation from page 6 and collecting all terms together now gives the result.

(3) If the characteristic is an odd prime r , the definition of induced primitive of order n now states that the coproduct of any such x will have a term $q^{(n-1)} \otimes [q^{(n-1)}]^{r-1}$. This factor of $r - 1$ now carries over to the relations from page 6 and, following the reasoning from the previous item in this lemma with that added $r - 1$ throughout, we obtain the result. \square

This lemma now permits the following result, relating the universal cobilinear coproduct to the Verschiebung and Frobenius.



Proposition 4.3 For any $x \in H_1 \widehat{\otimes} H_2$, we get

- (1) $\gamma(Fx) = (F \otimes F)(\gamma x)$
- (2) $(F \otimes 1)(\gamma(Vx)) = (1 \otimes V)(\gamma x)$
- (3) $(1 \otimes F)(\gamma(Vx)) = (V \otimes 1)(\gamma x)$

Proof (1) If $x = \phi(t)$, with $\phi : K \rightarrow H_1 \otimes H_2$ cobilinear (K connected) and $t \in K$, write ϕ_i for the composition of ϕ with projection on the i th component. We get:

$$\begin{aligned}\gamma(Fx) &= \gamma(x^p) = \gamma[(\phi(t))^p] = (\phi(t))^p = [\phi_1(t) \otimes \phi_2(t)]^p \\ &= [\phi_1(t)]^p \otimes [\phi_2(t)]^p = (F \otimes F)(\gamma x)\end{aligned}$$

- (2) Suppose now $x = \phi(q^{(n)})$, an induced primitive of order n (include here the case $n = 0$, viewing regular primitives as induced primitives of order 0) in characteristic r (including $r = 2$). Then $vx = \phi(q^{(n-1)})$. By the previous lemma, $\gamma(x)$ is of the form $\sum_{\alpha, \beta} \sum_{i=0}^n [q_\alpha^{(n-i)}]^{2^i} \otimes [[q_\beta^{(i)}]^{r-1}]^{2^{n-i}}$. We get then:

$$(1 \otimes V)(\gamma x) = \sum_{\alpha, \beta} \sum_{i=1}^n [q_\alpha^{(n-i)}]^{2^i} \otimes [[q_\beta^{(i-1)}]^{r-1}]^{2^{n-i}}$$

and

$$\begin{aligned}(F \otimes 1)(\gamma(vx)) &= (f \otimes 1) \left(\sum_{\alpha, \beta} \sum_{i=0}^{n-1} [q_\alpha^{(n-1-i)}]^{2^i} \otimes [[q_\beta^{(i)}]^{r-1}]^{2^{n-1-i}} \right) \\ &= \sum_{\alpha, \beta} \sum_{i=0}^{n-1} [q_\alpha^{(n-1-i)}]^{2^{i+1}} \otimes [[q_\beta^{(i)}]^{r-1}]^{2^{n-1-i}} \\ &= \sum_{\alpha, \beta} \sum_{j=1}^n [q_\alpha^{(n-j)}]^{2^j} \otimes [[q_\beta^{(j-1)}]^{r-1}]^{2^{n-j}}\end{aligned}$$

and the result follows.

- (3) The proof is similar to (2).

□

5 Cobilinear coproducts of Dieudonné modules

Next, we will do for connected Dieudonné modules a construction similar to the one carried out in the previous section. We define cobilinear maps for those Dieudonné modules, and introduce the notion of universal cobilinear coproducts in the same context. Dieudonné corings are defined, and we end with the equivalence of the categories of connected Hopf corings and connected Dieudonné corings.

As in Sect. 3, Hopf algebras will be considered in \mathcal{HA} , the category of ungraded connected Hopf algebras over \mathbb{F}_p , p a prime.

Recall that \mathcal{DM} was defined previously as the category of modules M over $R = \mathbb{Z}_p[V, F]/(VF = FV = p)$ such that for each $x \in M$ there exists an order $n \geq 0$ with $V^n x = 0$.

A *cobilinear map* for R -modules M , N and L is a map $g : M \rightarrow N \otimes L$ satisfying:

- (1) $g(Fm) = (F \otimes F)(gm)$
- (2) $(F \otimes 1)(g(Vm)) = (1 \otimes V)(gm)$
- (3) $(1 \otimes F)(g(Vm)) = (V \otimes 1)(gm)$

for every $m \in M$.

The *universal cobilinear coproduct* is defined (when it exists), for every pair of connected Dieudonné modules M_1 and M_2 , as the unique connected Dieudonné module $M_1 \widehat{\otimes} M_2$ together with a cobilinear map $\gamma : M_1 \widehat{\otimes} M_2 \rightarrow M_1 \otimes M_2$ that is universal with respect to all cobilinear maps. That is, such that for any

cobilinear map $K \rightarrow M_1 \otimes M_2$, there exists a unique Dieudonné module map $K \rightarrow M_1 \widehat{\otimes} M_2$ making the following diagram commute.

$$\begin{array}{ccc} M_1 \widehat{\otimes} M_2 & \xrightarrow{\gamma} & M_1 \otimes M_2 \\ \uparrow & \nearrow & \\ K & & \end{array}$$

Lemma 5.1 Any cobilinear map $g : DH \rightarrow DH_1 \otimes DH_2$, where H, H_1 and H_2 are connected Hopf algebras in \mathcal{HA} , induces a cobilinear map $g' : H \rightarrow H_1 \otimes H_2$.

Proof Since H is connected, it is enough to define g' on primitives and induced primitives [9].

Given a primitive $q \in H$, pick a positive m and consider $\tilde{q} \in \text{Hom}_{\mathcal{HA}}(H(m), H)$ given by $\tilde{q}(1) = 1$, $\tilde{q}(\omega_m) = q$ and $\tilde{q}(\omega_i) = 0$ for $i \neq m$ (Here, ω_i are the Witt polynomials.)

Then $g(\phi^m(\tilde{q}))$ is in $DH_1 \otimes DH_2$, and so the projections on DH_1 and DH_2 are such that

$$g_1(\phi^m(\tilde{q})) = \phi^r(\alpha) \text{ for some } r \text{ and some } \alpha \in \text{Hom}_{\mathcal{HA}}(H(r), H_1)$$

and

$$g_2(\phi^m(\tilde{q})) = \phi^s(\beta) \text{ for some } s \text{ and some } \beta \in \text{Hom}_{\mathcal{HA}}(H(s), H_2).$$

Define then $g'(q)$ as $\alpha(\omega_r) \otimes \beta(\omega_s)$.

If $q^{(n)}$ is an induced primitive (relative to the primitive q), we can still define \tilde{q} and obtain α and β as before.

$$\text{Put then } g'(q^{(n)}) = \alpha((\omega_r)^{(n)}) \otimes \beta((\omega_s)^{(n)}). \quad \square$$

For the next result, we construct a cobilinear map $\tilde{\gamma} : D(H_1 \widehat{\otimes} H_2) \rightarrow DH_1 \otimes DH_2$.

Start then with $\alpha \in D(H_1 \widehat{\otimes} H_2)$.

We have that $\alpha = \phi^m(x)$ for some $x \in \text{Hom}_{\mathcal{HA}}(H(m), H_1 \widehat{\otimes} H_2)$, where ϕ^m represents the map $\text{Hom}_{\mathcal{HA}}(H(m), H_1 \widehat{\otimes} H_2) \rightarrow D(H_1 \widehat{\otimes} H_2)$ given by the definition of colimit.

We get a map $\gamma^*x : H(m) \rightarrow H_1 \otimes H_2$, given by the composition of x with the map γ that defined the universal cobilinear coproduct in connected Hopf algebras. This map is a Hopf algebra map, as each $H(m)$ is polynomial (and connected), and because of the way γ was defined for connected Hopf algebras.

Projecting γ^*x in each component gives maps $H(m) \rightarrow H_1$ and $H(m) \rightarrow H_2$ and, by definition of colimit, we finally get $\tilde{\gamma} : D(H_1 \widehat{\otimes} H_2) \rightarrow DH_1 \otimes DH_2$ (The definition does not depend on the x picked initially.)

Note that this is a cobilinear map of Dieudonné modules, since γ was such a map for connected Hopf algebras and V and F are defined from the Verschiebung and the Frobenius by composition on the right.

Theorem 5.2 For any two connected Hopf algebras H_1 and H_2 in \mathcal{HA} , the connected Dieudonné module $D(H_1 \widehat{\otimes} H_2)$ is a universal cobilinear coproduct of the connected Dieudonné modules DH_1 and DH_2 .

That is, any DH_1 and DH_2 have a universal cobilinear coproduct and

$$(DH_1) \widehat{\otimes} (DH_2) = D(H_1 \widehat{\otimes} H_2).$$

Moreover, if $\gamma : H_1 \widehat{\otimes} H_2 \rightarrow H_1 \otimes H_2$ is the cobilinear map defining the universal cobilinear coproduct of H_1 and H_2 , then

$$\tilde{\gamma} : D(H_1 \widehat{\otimes} H_2) \rightarrow DH_1 \otimes DH_2$$

is the cobilinear map defining the universal cobilinear coproduct of DH_1 and DH_2 .

Proof Start with a connected Dieudonné module K and a cobilinear map

$$g : K \rightarrow DH_1 \otimes DH_2.$$

We know, by Theorem 3.2, that $K = DH$ for some connected Hopf algebra H .

$$g : DH \rightarrow DH_1 \otimes DH_2 \text{ induces a cobilinear map } g' : H \rightarrow H_1 \otimes H_2.$$



From the definition of universal cobilinear coproduct for connected Hopf algebras, there exists a unique Hopf algebra map $h : H \rightarrow H_1 \hat{\otimes} H_2$ such that

$$\begin{array}{ccc} H_1 \hat{\otimes} H_2 & \xrightarrow{\gamma} & H_1 \otimes H_2 \\ \uparrow h & \nearrow g' & \\ K & & \end{array}$$

commutes.

Applying the functor D , we get that $Dh : DH \rightarrow D(H_1 \hat{\otimes} H_2)$ is unique, and combining with $\tilde{\gamma}$ on the right finishes the proof. \square

Corollary 5.3 Any two connected Dieudonné modules have a universal cobilinear coproduct.

Proof The equivalence between categories from Theorem 3.2 states that any connected Dieudonné module M can be written as $M = D(UM)$, where UM is a connected Hopf algebra. \square

A connected Dieudonné coring will then be an ungraded connected Dieudonné module M together with a coproduct $M \rightarrow M \otimes M$ that is compatible with V and F .

From this last corollary we can get the following result, thus completing the picture on category equivalences in the case of connected Hopf corings.

Theorem 5.4 The category of connected Hopf corings is equivalent to the category of connected Dieudonné corings.

Proof This follows from the previous considerations, plus Theorem 3.3. and Proposition 4.3. \square

Acknowledgments This study was partially supported by the Fundação para a Ciência e Tecnologia (FCT/Portugal).

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

1. Bousfield, A.K.: On p -adic λ -rings and the K -theory of H -spaces. *Mathematische Zeitschrift*. **223**, 483–519 (1996)
2. Buchstaber, V.; Lazarev, A.: Dieudonné modules and p -divisible groups associated with Morava K -theory of Eilenberg–MacLane spaces. *Algebr. Geometr. Topol.* **7**, 529–564 (2007)
3. Goerss, P.G.: Hopf rings, Dieudonné modules and $E_*\Omega^2 S^3$. In: Meyer, J.-P.; Morava, J.; Wilson, W.S. (eds.) *Homotopy Invariant Algebraic Structures: A Conference in Honor of J. Michael Boardman*, volume 239 of *Contemporary Mathematics*, pp. 115–174 (1999)
4. Ravenel, D.C.: Dieudonné modules for abelian Hopf algebras. In: Davis, D. (ed.) *Reunión Sobre Teoría de Homotopía*, volume 1 of *Serie notas de matemática y simposia*, pp. 177–194. Sociedad Matemática Mexicana, Mexico, D.F. (1975)
5. Ravenel, D.C.; Wilson, W.S.: The Hopf ring for complex cobordism. *J. Pure Appl. Algebra* **9**, 241–280 (1977)
6. Sadofsky, H.; Wilson, W.S.: Commutative Morava homology Hopf algebras. In: Mahowald, M.E.; Priddy, S. (Eds.) *Homotopy Theory in Algebraic Topology*, volume 220 of *Contemporary Mathematics*, pp. 367–373, Prov (1998)
7. Saramago, R.M.: Dieudonné theory for ungraded and periodically graded Hopf rings. Ph.D. thesis, The Johns Hopkins University (2000)
8. Saramago, R.M.: Dieudonné module structures for ungraded and periodically graded Hopf rings. *Algebr. Represent. Theory*. **13**(5), 521–541 (2010)
9. Saramago, R.M.: Finite dimensional connected and geometric-like Hopf algebras. *Far East J. Math. Sci.* **34**(2), 189–205 (2009)
10. Schoeller, C.: Étude de la catégorie des algèbres de Hopf commutatives connexes sur un corps. *Manusc. Math.* **3**, 133–155 (1970)

